A 3D-Printed Ballistic Pendulum Retrofit for the Rotary Motion Sensor: Derivations of Equations 1 – 7

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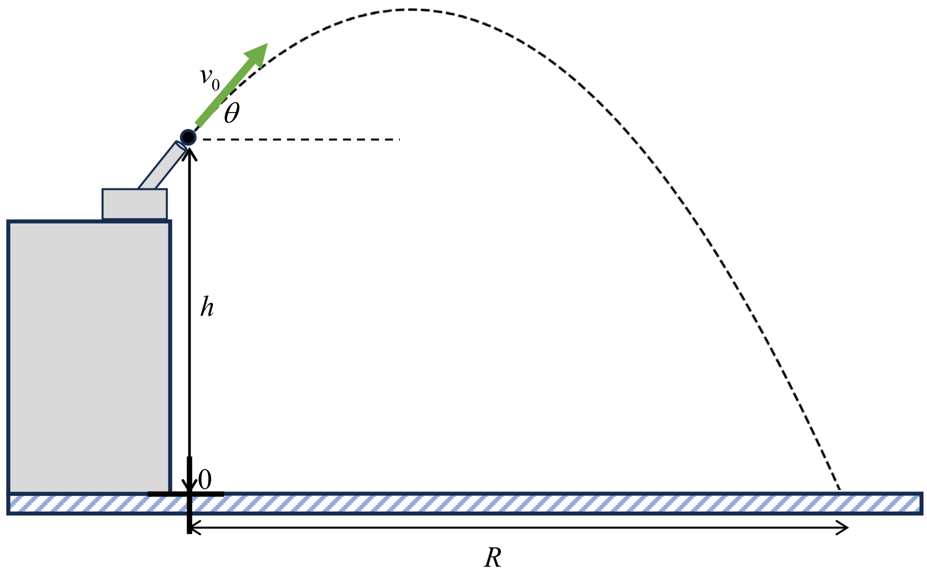
Eq. (1-2) Calculating the muzzle speed of a projectile launched from height h at initial angle θ relative to the horizontal with measured range R.



We place the origin directly below the projectile, at ground level. Starting with a horizontal analysis of the motion, we obtain an expression for flight time in terms of R, v_0 and θ:

$$x = \cancel{x_0} + v_{0x}t \Rightarrow R = v_0\cos\theta \cdot t \Rightarrow t = \frac{R}{v_0\cos\theta}$$

We proceed with a vertical analysis of the motion:

$$\cancel{y} = y_0 + v_{0y}t - \frac{1}{2}gt^2 \Rightarrow 0 = h + v_0\sin\theta\cdot t - \frac{1}{2}gt^2$$

Substituting our flight time from the horizontal analysis, we obtain:

$$0 = h + \cancel{v_0}\sin\theta\cdot\frac{R}{\cancel{v_0}\cos\theta} - \frac{1}{2}g\left(\frac{R}{v_0\cos\theta}\right)^2 \Rightarrow 0 = h + R\tan\theta - \frac{gR^2}{2{v_0}^2\cos^2\theta}$$

And finally, solving for v_0:

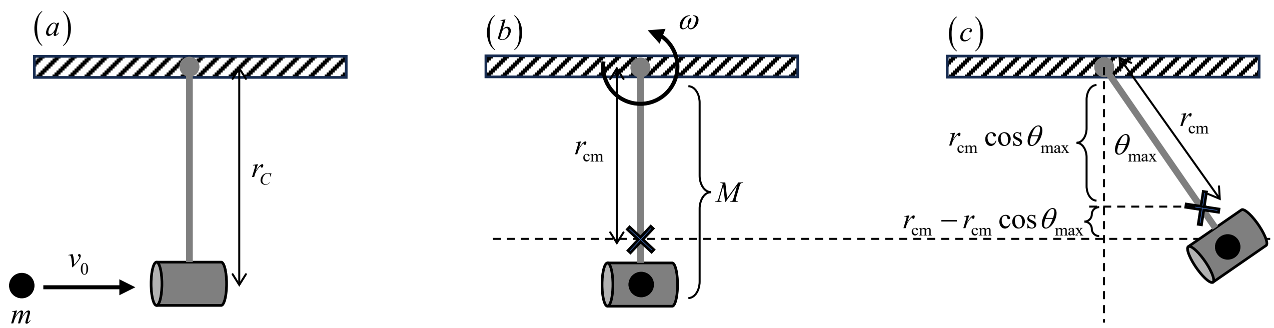
$$\Rightarrow \frac{gR^2}{2{v_0}^2\cos^2\theta} = h + R\tan\theta \quad \Rightarrow \quad 2{v_0}^2\cos^2\theta = \frac{gR^2}{h + R\tan\theta} \quad \Rightarrow \quad \boxed{v_0 = \sqrt{\frac{gR^2}{2\cos^2\theta\left(h + R\tan\theta\right)}}} \quad \text{Eq. (1).}$$

To find the muzzle speed from a horizontal launch, we simply substitute zero for θ to obtain

$$\boxed{v_0 = R\sqrt{\frac{g}{2h}}} \quad \text{Eq. (2)}$$

Eq. (3): Finding the initial speed of a ball using the deflection of a ballistic pendulum.

Below, we show three phases of the process with appropriate labels:



In (a) our projectile of mass *m* is moving with speed v_0 prior to impact. We also label r_C, the distance from the rotation axis to the center of the catcher.

In (b), the impact has just occurred, and the combined masses (total *M*) are moving with an angular velocity of ω about the pivot point. Note that the center of mass of the system has been marked with an X, and the distance from the rotation axis to center of mass of the combined masses, r_{cm}, has been labeled in this diagram.

In (c), the pendulum has reached its maximum angle of deflection θ_{max}. We include a horizontal dashed line as a reference point to the original vertical position of the center of mass, because we need to measure the change in height of the center of mass for the gravitational potential energy calculation. We took the liberty of working out the trigonometry in the diagram: we show the vertical projection of r_{cm} ($r_{cm}\cos\theta_{max}$), making the change in height of the center of mass $r_{cm}-r_{cm}\cos\theta_{max}$, or $r_{cm}\left(1-\cos\theta_{max}\right)$.

We now proceed with the conservation of angular momentum/conservation of energy analysis.

In the initial impact, angular momentum should be conserved, because the ideal hub can exert no external torque on the system, and the vertical position of the pendulum means that gravity exerts zero external torque as well. The initial angular momentum of the ball is mv_0r_C, so the angular momentum analysis proceeds as follows:

$$L_i = L_f \;\;\Rightarrow\;\; mv_0r_C = I\omega \;\;\Rightarrow\;\; v_0 = \frac{I\omega}{mr_C} \;\; (i)$$

In the next phase, the rotational kinetic energy of the combined masses is converted to gravitational potential energy as the pendulum rises to its maximum height:

$$E_i = E_f \;\;\Rightarrow\;\; \frac{1}{2}I\omega^2 = Mgr_{cm}\left(1-\cos\theta_{max}\right) \Rightarrow \omega = \sqrt{\frac{2Mgr_{cm}\left(1-\cos\theta_{max}\right)}{I}} \;\; (ii)$$

Substituting Eq. (*ii*) into Eq. (*i*), we obtain:

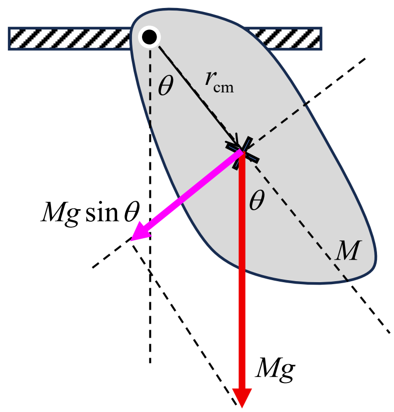
$$\Rightarrow v_0 = \frac{I}{mr_C}\sqrt{\frac{2Mgr_{cm}\left(1-\cos\theta_{max}\right)}{I}}$$

, which simplifies to $\boxed{v_0 = \frac{1}{mr_C}\sqrt{2MIgr_{cm}\left(1-\cos\theta_{max}\right)}}$ Eq. (3).

Eq. (4): Finding the moment of inertia from the period of small oscillations of a physical pendulum.

Finding the period of oscillations of a physical pendulum is a standard textbook example, but we repeat it here for your convenience.

Below, we show a physical pendulum rotating about a hinge, with the center of mass position vector inclined at an angle of θ relative to the vertical (the center of mass is marked with an X). We have already taken the liberty of putting the weight vector into the diagram (acting at the center of mass) and finding its tangential component for the torque calculation:



Now we can apply the "rotational equivalent of Newton's 2nd law", $\tau = I\alpha$, where τ is the torque exerted by gravity and α is the angular acceleration.

$$\tau = I\alpha \quad \Rightarrow \quad -Mgr_{\text{cm}} \sin\theta = I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2},$$

where we are viewing a positive angle in our diagram (defined as the counterclockwise direction) and we insert the minus sign to indicate the direction of torque (clockwise). We have also replaced the angular acceleration with a second time derivative of θ, so we are looking at a non-linear 2nd order ODE, which cleans up a bit as

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{Mgr_{\text{cm}}}{I}\sin\theta .$$

To linearize this differential equation, we write down the power series expansion of the sine function

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots,$$

and we make a "small angle" approximation to justify truncating this series after the linear term to obtain $\sin\theta \approx \theta$.

Note that we ran the numbers today for an angle of 15° or $\frac{\pi}{12}$ radians and obtained about a 1% difference between $\sin\theta$ and θ, and of course the approximation becomes more accurate as the angle becomes smaller. Make a note of this for the lab procedure – you don't want students using enormous angles!

The whole point of linearizing our differential equation is that the resulting equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{Mgr_{\mathrm{cm}}}{I}\theta$$

can be solved by inspection* as

$$\theta(t) = c_1 \sin\left(\sqrt{\frac{Mgr_{\mathrm{cm}}}{I}} \cdot t\right) + c_2 \cos\left(\sqrt{\frac{Mgr_{\mathrm{cm}}}{I}} \cdot t\right).$$

This solution has a period of 2π over the coefficient of *t*, and we obtain the period of the physical pendulum

$$T = 2\pi\sqrt{\frac{I}{Mgr_{\mathrm{cm}}}}\ .$$

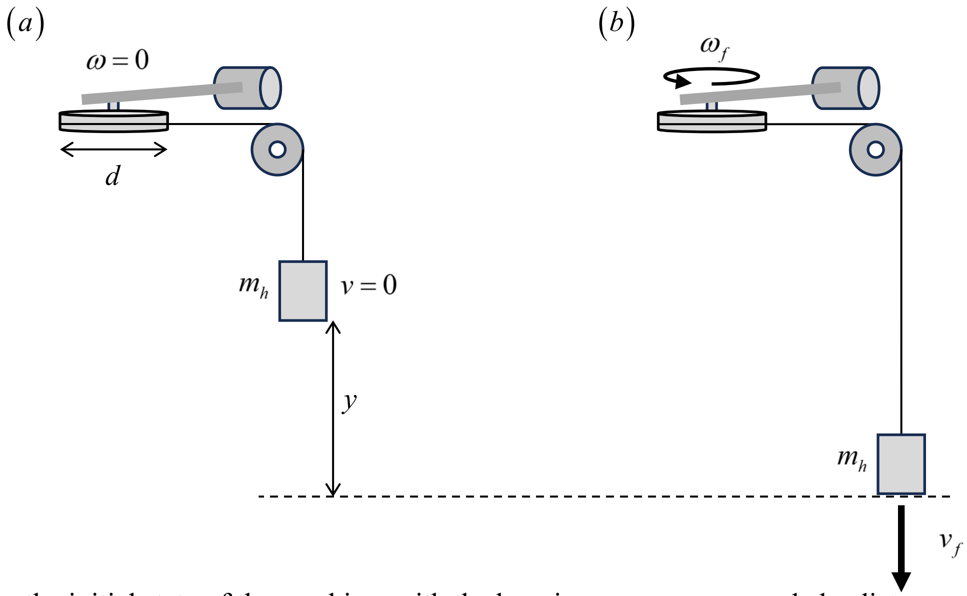
Now we just have to solve for *I*:

$$T = 2\pi\sqrt{\frac{I}{Mgr_{\mathrm{cm}}}} \Rightarrow \left(\frac{T}{2\pi}\right)^2 = \frac{I}{Mgr_{\mathrm{cm}}} \Rightarrow \boxed{I = \frac{Mgr_{\mathrm{cm}}T^2}{4\pi^2}} \quad \text{Eq. (4)}$$

*This is a classic oscillatory differential equation of the form $\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} = -\omega^2 f$; i.e., a simple harmonic oscillator. To guess the solutions, we just need to think of functions that we can differentiate twice to obtain the original function together with a minus sign and a constant out in front. These are sines and cosines, and we deal with the constant ω by considering two iterations of the chain rule. Thus, the general solution (a linear combination of linearly independent solutions) is given by $f(t) = c_1 \sin(\omega \cdot t) + c_2 \cos(\omega \cdot t)$.

Eq. (5): Finding the moment of inertia by making a "rotational Atwood machine"

In the diagram below, we show the initial and final state of the "rotational Atwood machine" setup, with all appropriate quantities labeled in the diagrams:



In (a) we show the initial state of the machine, with the hanging mass m_h suspended a distance y above its final position. Note that the string is wrapped around the hub of diameter d here.

In (b) we show the final state of the machine, with the hanging mass arriving at a height of zero and the pendulum/hub assembly rotating with final angular velocity ω_f. Note that we will not neglect the final kinetic energy of the hanging mass (even though in practice it's pretty small), so we include a final velocity of v_f in the diagram.

Before we begin with the conservation of energy analysis, we note that the final angular speed ω_f is related to the final linear speed of the mass v_f by the formula $v_f = \frac{d}{2}\omega_f$ (*i*) (this is just the usual $v = r\omega$ for the speed at the edge of a rotating disk, which is the same as the speed of the string and therefore the same as the speed of the hanging mass).

We apply conservation of energy, and this time we have gravitational potential energy transforming to the rotational kinetic energy of the pendulum assembly in addition to the linear kinetic energy of the falling mass:

$$E_i = E_f \quad \Rightarrow \quad m_h g y = \frac{1}{2} I \omega_f^2 + \frac{1}{2} m_h v_f^2$$

Substituting Eq. (*i*) for v_f, and solving for *I*, we obtain:

$$m_h g y = \frac{1}{2} I \omega_f^2 + \frac{1}{2} m_h \left(\frac{d}{2}\omega_f\right)^2 \quad \Rightarrow \quad 2 m_h g y = I \omega_f^2 + m_h \frac{d^2}{4}\omega_f^2 \quad \Rightarrow \quad I \omega_f^2 = 2 m_h g y - m_h \frac{d^2}{4}\omega_f^2$$

$$\Rightarrow \boxed{I = m_h \left(\frac{2gy}{\omega_f^2} - \frac{d^2}{4} \right)} \quad \text{Eq. (5)}$$

Eq. (6): the piecewise moment of inertia of the ballistic pendulum

While this is also standard textbook material, we decided to include the derivation for your convenience (and ours – it was simpler to just state "all formulas in this paper have full derivations in the research notes"!).

Recall the basic definition of the moment of inertia of a collection of point masses:

$$I = \sum_i m_i r_i^2$$

where the m_i's are the masses of the particles and the r_i's are their respective distances from the rotation axis.

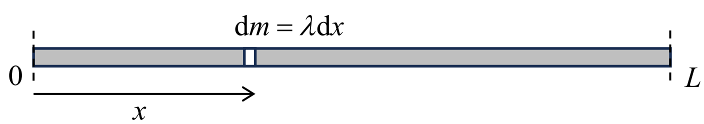
We can always break one collection of particles into many sub-collections, so we see immediately that moments of inertia for a composite object simply add.

This allows us to find the contribution due to the catcher (which we are just modeling as a point mass):

$$I_{catcher} = m_C r_C^2 .$$

As for the rod rotating about its end, we have to do a standard physical integral to find the moment of inertia of a continuous object, but the same basic idea applies: we break the object into point masses and add up the contributions to the moment of inertia.

In the diagram, we show a rod of mass M and length L located on the positive x-axis with its left end at the origin. We also label a mass increment $\mathrm{d}m$ at a location of x. $\mathrm{d}m$ can be calculated as $\mathrm{d}m = \lambda \mathrm{d}x$, where λ is the linear density of the rod and $\mathrm{d}x$ is the infinitesimal increment of length containing $\mathrm{d}m$:



Now the moment of inertia contribution of our point mass is simply $\mathrm{d}I = x^2 \mathrm{d}m = \lambda x^2 \mathrm{d}x$, and we're ready to sum the point masses:

$$I = \int \mathrm{d}I = \lambda \int_0^L x^2 \mathrm{d}x = \frac{1}{3}\lambda L^3 .$$

Replacing the linear density with M/L for this uniform rod, we obtain

$$I = \frac{1}{3}\frac{M}{L}L^3 = \frac{1}{3}ML^2 ,$$

the standard formula for moment of inertia of a rod rotating about its end.

Finally, placing our total moment of inertia in the notation used in our paper, we get

$$\boxed{I = m_C r_C^2 + \frac{1}{3} m_R L_R^2} \qquad \text{Eq. (6)}$$

Eq. (7): finding the center of mass of a composite object from the CM of each piece

Finally, a useful, but minor, note on calculating the center of mass of a composite object.

Recall that the center of mass of a collection of particles is defined as

$$\vec{r}_{cm} = \frac{1}{M}\sum m_i \vec{r}_i ,$$

where the m_i 's are the masses of the particles, the $\vec{r}_i$'s are the position vectors for all the particles and M is the total mass of all the particles.

This quantity pops up in many of the derivations of the theorems of classical mechanics, and we often find it useful to recognize the statement in this form:

$$M\vec{r}_{cm} = \sum m_i \vec{r}_i \quad (i)$$

Now we consider a composite object made of two pieces, *A* and *B*, and write down the center of mass position vector:

$$\vec{r}_{cm} = \frac{1}{M}\sum m_i \vec{r}_i = \frac{1}{M}\left(\sum_A m_i \vec{r}_i + \sum_B m_i \vec{r}_i\right),$$

Where we are simply summing over the particles in object *A* and *B* separately in order to sum over all the particles. Now we recognize the right-hand side of Eq. (*i*) in our summations: each of the sums is simply a total mass multiplied by a center of mass position vector:

$$\vec{r}_{cm} = \frac{1}{M}\left(M_A \vec{r}_{cm_A} + M_B \vec{r}_{cm_B}\right).$$

In other words, we can find the center of mass of a composite object by treating it as a sum of point masses, each located at its respective center of mass. Translating to a 1-D setting and using the notation in our paper, we arrive at

$$\boxed{r_{cm} = \frac{1}{M}\left(m_C r_C + m_R r_R\right)} \quad \text{Eq. (7)}$$